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Forms of Generalization in Students Experiencing Mathematical Learning Difficulties

George Santi and Anna Baccaglini-Frank

We shift the view of a special needs student away from the acknowledged view, that is as a student who requires interventions to restore a currently expected functioning behaviour, introducing a new paradigm to frame special needs students' learning of mathematics. We use the theory of objectification and the new paradigm to look at (and characterize) students' learning experiences in mathematics as generalizing reflexive mediated activity. In particular, from this perspective, we present examples of shifts to higher levels of generalization of a student with mathematical learning difficulties working with Mak-Trace, a Logo-like educational software for the iPad.

Keywords: Mak-Trace; Mathematical generalization; Mathematical learning difficulties; Special needs students

Formas de generalización en estudiantes con dificultades de aprendizaje en matemáticas

En este artículo introducimos un nuevo paradigma para enmarcar el aprendizaje de las matemáticas de alumnos con necesidades especiales. Consideramos una visión de los estudiantes con necesidades especiales diferente a la comúnmente aceptada que los considera como estudiantes que requieren intervención para reestablecer el comportamiento actualmente esperado. Utilizamos la teoría de la objetivización y ese nuevo paradigma para observar (y caracterizar) las experiencias de aprendizaje de las matemáticas entendido como actividad reflexiva y mediada de generalización. En particular, desde esta perspectiva proponemos ejemplos de acceso a niveles superiores de generalización de un estudiante con dificultades de aprendizaje de las matemáticas que utiliza Mak-Trace, un software didáctico para iPad parecido a Logo.

Términos clave: Dificultades de aprendizaje en matemáticas; Estudiantes con necesidades especiales; Generalización matemática; Mak-Trace

Santi, G., & Baccaglini-Frank, A. (2015). Forms of generalization in students experiencing mathematical learning difficulties. *PNA, 9*(3), 217-243.

Generalization

From a social and cultural point of view, mathematics is perceived as the subject that develops logical-deductive reasoning and abstraction as well as providing students with the language of science and technology. Underneath logical-deductive reasoning and abstraction lays a constitutive feature of human thinking and action, generalization. In fact effective logical deductive reasoning, for example a deduction, an induction or an abduction, allows to grasp a property that applies to all cases. Our action, as mathematicians, is effective when we can apply procedures that solve a broad set of problems.

A thorough analysis of this topic would require a philosophical overview that starts from ancient philosophy (Plato and Aristoteles) up to contemporary philosophy, and it is beyond the scope of this paper. We will touch on a few benchmarks of such philosophical overview. The main cause of students' difficulties in generalizing can be re-conducted to Duval's argument that mathematical objects are ideal and inaccessible (1999). Therefore, as highlighted by Radford (2004), the ontology of mathematical entities are connected in generalization. How can the individual experience in the learning process that which is intrinsically inaccessible? This becomes immediately tangible when, for example, we introduce variables to our students and tell them that the letter stands for any number. Students are often unable to overcome treating the letter as a specific entity that they can manipulate, thereby triggering all the undesired behaviours broadly described by researches on the didactics of algebra (see, for example, Kieran, 1989; Küchemann, 1981; Küchemann & Hoyles, 2009).

In a purely rationalist conception of cognition, generalization is subsumed in the activity of the mind, what Descartes would call the *res cogitans*. The particular belongs to the realm of the *res extensa*, whose reality and action is driven not by perception but by reason. Kant in the Critics of Pure Reason questioned the basis of the subject's possibilities to know, synthesizing rationalism and empiricism. The philosopher conceives the issue of generalization as the relation between *a priori* concepts and our sensible experience; knowledge is within this relationship which is the only possible access to the *per se* object (the *noumenon*) as a phenomenon.

As for mathematics, Kant considers the a priori concepts the general, and the sensible experience the particular. He establishes a relationship between the general and the particular. According to Kant the general and the particular join together in the schema. Through the schema, the mathematical *a priori* concept descends into the world of sensible experience without losing its essential characteristics, i.e. generality. When we draw a rectangle, we have a sensible experience of this geometrical entity. The drawing is a particular case that betrays the generality and the a priori nature of the concept. Nevertheless, the essence of the concept is untouched in the operational invariant of the schema that allows us to draw it and is always beyond the drawing. Fischbein's (1993) notion of figural concept can be interpreted as a paradigmatic example of this tension between general and particular. The importance of schemas in mathematical practice is strictly related to the role of and confidence in signs as bridges between sensible experience and a priori concepts. We owe to Kant the merit of ascribing an epistemological role to signs (Radford, 2004).

Kant's epistemology is, however, basically essentialist, failing to take into account cultural and historical experience as constitutive of mathematical concepts and their ideality. Within a socio-cultural perspective, Radford's theory of objectification (TO) bridges the gap between the ideality of mathematical objects and cultural and historical experience. Radford (2008) considers mathematical objects conceptual forms of historically, socially, and culturally embodied, reflective, mediated activity (Radford, 2006). The generality of mathematical objects is consubstantial with, and is derived from human activities. Ilyenkov (1977) clarifies the cultural and historical origin of generality.

> *"Ideality" is rather like a stamp impressed on the substance of nature by social human life activity, a form of the functioning of the physical thing in the process of this activity. So all the things involved in the social process acquire a new 'form of existence' that is not included in their physical nature and differs from it completely—[this is] their ideal form.* (p. 86)

The tension between general and sensible experience is still vibrant in this approach. Such tension is not between a priori concepts and sensible experience but between the individual's sensible experience and the reflexive mediated activity condensed in the ideality of the historical and cultural object. Within a socio-cultural perspective, the study of generalization requires taking into account not only the ontological and epistemological dimensions but also the anthropological and socio-cultural ones. This point will be crucial when analyzing generalization processes of special needs students, as we set out to do in this paper.

Special Needs Students

According to the pedagogical and socio-cultural point of view, school systems around the world have basically taken two different approaches to inclusion of special needs students in the educational system. One option is to create special schools that are intended to include students with a same deficit. The objective is to provide the best teaching environment specific for that class of students, in an attempt to help them to overcome their handicap and to avoid their being left behind. Another option is to include special needs students in regular schools, granting them specific support in terms of teaching, technical devices and selection of appropriate learning objectives. This choice is inspired by the ethical and pedagogical principle that special needs students have the right to attend school with their peers and be part of the social and cultural life of their nations' educational system. Italy has adopted the second option, with a declared intention (see, for example, Ministerio dell'Istruzione dell'Università e della Ricerca [MIUR], 2012) of fostering the social inclusion of special needs students in the life of the school and of their peers. Nevertheless, in reality, these students are not always included and they do not actually play a key role (or any role, at that!) in the teaching-learning process.

Both approaches conceive the special needs student as a cognitive system with some form of temporary or permanent dysfunction that hinders, as regards education, the possibility to achieve the learning objectives outlined by the school system. The school systems have to, therefore, provide the most effective instruments and organization of the teaching processes that allow the student to reach specific objectives, possibly selected among the ones of the "normal" students. Current special needs students' policies still conceive thinking from a rationalistic standpoint that reduces cognition to the activity of the brain and of its recognized processes. Special education aims at creating the best conditions to bring disabled students as close as possible to the objective of the so-called "normal" students. There is no focus on the way the student actually experiences mathematics in terms of perception, movement, feelings, or on the social and cultural nature of mathematics both in its historical development and in the teaching and learning process. Although the aim of special education is to include all students, it is often the case that the intent of teaching is to try to conform disabled students to the standards expected for normal students (on this theme and on the inadequacy of this approach also see Vygotsky's "The Fundamental Problems of Defectology" (1929).

Canevaro (1999), an Italian educator to whom the community owes so much for his forefront research in special education and for his commitment to the inclusion of special needs students in schools, makes the distinction between disability and handicap. A disability is intrinsic and unavoidable whereas the handicap is due to the environment. The notion of handicap is generally understood in terms of a functional obstacle the disabled person encounters because of his/her environment. Typically, the student on a wheelchair suffers a handicap when s/he cannot reach the school premises because there are no facilities. In this paper we will use special needs students as a synonym of disabled students, and we argue that they should not become handicapped.

We believe it is necessary to go beyond the dichotomy between special needs/normal students. Although in principle we agree that taking cognitive deficits (from a clinical point of view) into account can be (and may be more so in the future) useful to support the student appropriately, from an educational point of view—at our current state of knowledge—this distinction can be misleading. All students *have to* face difficulties, learning obstacles and failure throughout their education, therefore it is not clear how to identify a clear boundary between normal and disabled students. Sfard (2014) claims that there is really no distinction between normal and disabled students. She claims that brains can have different proficiencies that lead some students to be slower, possibly more holistic, with a more aesthetic mood, more embodied, etc. Sfard highlights how the learning of new knowledge starts with incomprehension, for all students! Therefore the question should not be "how to account for the fact that some children have difficulties and some don't" but rather "how to account for the fact that some students can overcome difficulties [and some don't seem to be able to do so]" (Sfard, 2014). This opens a social perspective on teaching and learning that allows to go beyond the individual dimension.

When facing the difficulties related to a new topic, students perform mathematics routines as rituals and through discipline, confidence in the teacher and in their abilities, and the functions of their brain they transform such rituals into exploration and awareness of the mathematical meaning. In the presence of a special needs student some functions of the brain can be compromised but discipline and confidence can be fostered by the teaching action and by the environment to which the student is exposed. Therefore all students encounter difficulties in the learning of mathematics and the role of teaching and of the environment can be the key factor for successful learning, also in the case of difficulties. Sfard (2014) outlines two basic elements that hinder all students in overcoming the difficulties intrinsic to the learning of mathematics. The first has to do with the system of beliefs that the teacher and the environment lead the student to develop: This usually results in an identity of failure. For a student who manifests difficulties with mathematics this can be really serious, compromising his/her identity and psychological evolution (also see Heyd-Metzuyanim, 2012).

The second factor has to do specifically with teaching strategies that can expose the student too early to mathematical concepts or expect a rate of progress that does not respect his/her needs, attitudes and feelings. It can also happen that teaching fails to match the mathematical meaning with the individual meaning ascribed to mathematics through the students' needs, questions and activity.

All in all, we see the distinction between normal and special needs students as *naïve*. For all students education in general and special education in particular is an issue of identity, of the student's identity that has a social, cultural and historical nature. In the presence of disabilities teaching should be flexible and sensitive enough to look for the social, cultural and historical elements that contribute to the identity of the student.

Towards a New Paradigm

The aim of this section is to provide a new paradigm that frames teaching and leaning processes involving special needs students, going beyond the narrow dichotomy needs/normal students, instead resorting to different strategies according to the peculiarities of each subject. Such a change of paradigm aims at overcoming a contradiction in special education. The teaching activity strives to allow the disabled student to reach as much as possible, according to his/her possibilities, the same objectives of normal students, thereby disregarding his/her identity and being special from many points of view (cognitive, social, communicative, emotional, perceptive…). The stand point behind this approach is that thinking and learning is purely in the functioning of the mind (or, according to neurosciences, in the brain) and that a deficit provokes a dysfunction that has to be recovered resorting to a variety of supports: technological, didactical, psychological, and social. In this respect, the insight potentially brought to the classroom by the special needs student is not taken into consideration, but instead crushed in the attempt to homogenize all students' contributions.

In order to outline the new paradigm, we need to shift our focus from the analysis of cognitive functioning to the understanding of the experiences lived by students when learning mathematics. In presence of disabilities we must understand the kind of experiences lived by the student to achieve meaningful learning of mathematics that respects his/her needs and potentiality. There is no "dysfunctioning" that has to be "recovered" in order to allow the student to be included in the curriculum devised for normal students and keep up with his/her peers.

Proposed paradigm shift consider that

> *educational activity should aim at fostering a mode of existence in mathematics, i.e. being and becoming with others to make sense of the world also through mathematics. The aim of education should be to allow all students to make sense of the world in spite of their particular conditions.*

To investigate all students'—including special needs students'—difficulties faced in learning as a mode of existence that allows to make sense of the world we resort to Radford's theory of knowledge objectification (TO).

The TO is a Vygotskyan social cultural semiotic theory of mathematics thinking and learning that can be considered as a science of the subject (Roth & Radford, 2011) conceived within Leont'ev cultural-historical activity theory (1978). Individual consciousness, is conceived beyond an individual inner space by taking into account the constitutive role of activity culturally mediated by artifacts. According to the TO, consciousness is a cultural and social form of reflection that attends to a world of phenomena. When investigating teaching-learning processes the TO does not look at what is "inside" the mind in terms of structures, functions and contents but it examines how the individual, through activity, culturally makes sense of the world directing his/her intention to a cultural world of phenomena.

After providing brief descriptions of the basic constructs from the TO that we will use to analyze a few episodes from a teaching intervention with a special needs student learning mathematics, engaging, in particular, in generalization activities. This framework will allow us to advance hypotheses that attempt to account for the fact that the student doe not seem to be able to overcome particular difficulties, seen as steps in a process of generalization.

Mathematical Thinking and Mathematical Objects

Mathematical thinking is seen as a "mediated reflection in accordance with the form or mode of activity of individuals" (Radford, 2008, p. 218). Mediation means that, in a pragmatic view in continuity with Vygotsky's (1962) path, signs are constituents of thinking because they carry out the social activity and bind the individual to historical and cultural dimensions. Such mediators include sign systems, objects, instruments, gestures, etc. The reflexivity of thinking regards the role of subjective consciousness in thinking. The term reflection refers to the manner in which the individual intends reality according to cultural and social criteria. The TO defines mathematical objects as “fixed patterns of reflexive activity incrusted in the ever changing world of social practice mediated by artefacts” (Radford, 2008, p. 222).

Learning: Objectification and Semiotic Means of Objectification

Learning is considered a mediated reflexive activity but addressed towards the mathematical objects that bare a cultural and historical dimension. Learning is an active process of understandings and interpretations that allows to make sense of mathematical objects. To learn, then, is to objectify something. Radford terms such process objectification (2005, p. 111).

Learning as a process of objectification is a social reflexive activity that directs the individual’s intentional acts towards a cultural mathematical object through a set of semiotic resources that are the semiotic means of objectification (SMO). They include natural language (oral and written), gestures, objects, artefacts, bodily actions, and mathematical symbolic language. SMO are “bearers of an embodied intelligence and carry in themselves, in a compressed way, cultural-historical experience” (Radford, 2006). Learning is an intentional act in which the subject encounters and puts in “front” of his consciousness the mathematical object through a mediated activity that gives a cultural sense to the learned object: Language, signs, and objects are bearers of an embodied intelligence and carry in themselves, in a compressed way, cultural-historical experiences of cognitive activity (p. 52).

Generalization and Meaning

In the TO generalization is intrinsic to the process of objectification: Since thinking and learning are forms of reflection, the way we reflect the ideal cultural object entails the level of generality with which we attend to it. According to the way SMO allow us to intend the mathematical object, Radford (2003, 2004) recognizes three forms of generalization.

Factual generalization. Factual generalization is bound to operational schemas within the students’ space-time embodied experience, using rhythm, bodily movements, gestures, the generative and deictic use of natural language and working on specific objects.

Contextual generalization. Contextual generalization is bound to an invariant operational schema that keeps memory of space-time contextual experience without referring to a particular representation of the object that is objectified as conceptual object using linguistic deictic and generative terms, “this”, “above”, “below”, “next” “always”.

S*ymbolic generalization.* Symbolic generalization requires to drop the relation with contextual space-time elements using symbolic SMO.

Also intrinsic to objectification is the meaning of mathematical objects: Objectifying is a sense-making process that unravels as the student becomes aware of mathematical knowledge in the phenomenology of his/her consciousness when s/he directs his/her intentional acts towards the cultural object. The sense making process on the part of the student to become aware of the mathematical object endows meaning with a dual nature: The subjective content as intended by the individual's intentions and the cultural construct endowed with cultural values and theoretical content Radford (2006, p. 53). The sense-giving activity students are involved in can be seen as a convergence of the cultural meaning with the personal meaning. At an ontogenetic level the personal activity mediated by the SMO traces out the phylogenetic activity culturally condensed in the mathematical object.

The objectification process entails mainly three difficulties for the student.

- The mathematical object is an entity stratified in layers of generality. Each layer of generality is associated with a particular reflexive activity determined by the characteristics of the SMO that mediate it. The diversity of the student's reflexive activities splits his/her intentional acts towards objects that s/he considers disconnected but at an interpersonal level are recognized as belonging to the same cultural entity. The objectification process therefore requires a coordination of the different activities mediated by the different SMO.
- Meaning has a strongly embodied nature related to the personal space-time sensual and emotional experience of the student, but at higher levels of generality the student has to attend to the mathematical object also resorting to formal and abstract symbols that brake the relationship with his/her spatial and temporal sensorimotor experience. Students have to experience a disembodiment of meaning that hinders the objectification of the interpersonal dimension and the generality of the mathematical object.
- Students can experience a contradiction between the personal meaning, as lived first person experience, and the cultural meaning. The cultural meaning condensed in the SMO can mediate an activity that hinders the student's sense of proximity and consciousness' movement towards the generality of the mathematical object that transcends him/her.

Objectification for Special Needs Students

When teaching special needs students we have to take into account, in the process of objectification, the role of disabilities that modify and determine the reflexive mediated activity. The ambit of consciousness intertwined with activity is basically made of feelings, perceptions, movement, language and discourse, communication, relation with others. A disability can sensibly modify the nature of the reflexive activity of a special needs student by compromising or changing one of the elements mentioned above: perception (blind, deaf…), movement (brain and spine damage), language and discourse (dyslexia, dyscalculia, autism, schizophrenia…), communication (autism, schizophrenia, psychological disabilities), feelings and relationships (psychological disabilities, schizophrenia, social and cultural deprivation of the family context…).

We would like to draw the reader's attention to the fact that we are considering the disability only from a clinical point of view. From an educational one, we look at disabilities to understand the modes of existence available to the student in terms of feelings, perceptions, movement, communication and relationships when s/he objectifies mathematical knowledge. The special needs student has to face the same difficulties listed at the end of Section 2. Like other students, s/he faces the same problems related to generalization when having to recognize the same cultural object stratified in layers of generality, when s/he lives the disembodiment of meaning at higher levels of generality, and when s/he experiences a contradiction between his/her personal meaning and the cultural-interpersonal one. Also the teacher has to face the same didactical problems that s/he has to face with the other students: Exposing all students to and fostering meaningful activities and managing effectively the sociocultural space of the classroom in terms of SMO and cultural modes of signification.

There are three main issues that guide our analyses of the objectification of special needs students. It would be senseless both to try to "erase" the disability with some form of compensation—technical, didactical or psychological—or selecting from the curriculum mathematical knowledge the student "will be able to reach".

- The first issue postulates the possibility of identifying the SMO suitable for a special needs student safeguarding his emotional well-being during the reflexive activity. In the experimentation we will present in the following section the shift of SMO changes the emotional state of the student who becomes engaged with the reflexive activity.
- The second issue addresses the levels of generalization we can achieve with a disabled student. Our concern regarding generalization does not have to do with the student's success in the learning of mathematics but with the very nature of learning as objectification, his/her possibility of making sense of the mathematical objects at a certain level of generality.
- The third issue has to do with the alignment between personal and cultural meaning. By exposing students to suitable activities mediated by the SMO that bear the cultural and historical experience that lead to the emergence of the intended mathematical object, they can align their personal meanings with the cultural ones. If the special needs student cannot access those SMO s/he cannot overcome the distance between the cultural meanings and the personal ones that derive from the reflexive activity mediated according to his/her needs and possibilities. For example a blind student cannot objectify geometry using sight and drawings. If this distance remains, education fails in including the special needs student in the appropriate cultural and historical dimension.

Going back to Canevaro's considerations on handicap versus disability (1999), we believe that the cultural exclusion disabled children undertake has the nature of a handicap rather than of an intrinsic disability. The special needs student with his/her mode of existence, lived through the mediated reflexive activity brings a new culture that often is not recognized by the teacher and his/her peers. The disabled student brings into the classroom new strategies, ways of perceiving and feeling mathematics, often new technology. Instead of aligning the personal meanings of the special needs student with the cultural meanings outlined by the teacher, special education should trigger a dialogue between the cultural-historical dimension of mathematics with the new culture of the special needs student.

A Teaching Experiment: The Case of Filippo

In this section we outline a teaching intervention that was carried out between a special education teacher (the first author) and a student of his, using a particular Logo-like microworld.

Our Methodological Choice

Why did we choose to work in this type of an environment? Logo was designed around two principles that are: The importance of making mistakes and correcting oneself, and the student's taking responsibility of the construction of his/her knowledge, because teaching the computer to do something presumably teaches oneself more about their own thinking (Papert, 1980). Although not much research has been conducted on the use of Logo and/or of Logo-like environments in teaching students with special needs, research has suggested that it does make sense to use such environment with students with mathematical learning difficulties because it allows to operate with fundamental educational goals such as the following.

> *The student can: Establish a goal, remain absorbed in a task for a period of time, initiate communication about an idea or discovery, ask questions about content (not procedures), tolerate a period of confusion (with appropriate support) during new learning, try a new strategy when one has failed, use errors as a source of information about what to try next.* (Russell, 1986, p. 103)

Moreover, Vasu, and Tyler (1997) state that such environments may help develop spatial abilities and critical thinking skills.

Other researchers have reported several potential benefits of using Logo with students who have learning differences (Atkinson, 1984; Maddux, 1984; Michayluk & Saklofske, 1988; Miller, 2009; Ratcliff & Anderson, 2011). In particular, as previous research has suggested, using a more structured, mediated approach seemed to promote more purposeful behavior and greater understanding of concepts.

Microworlds are usually interpreted as a space of interaction between the student and the affordances of the environment that activate the cognitive development of the individual. Instead, with respect to the aim of this paper and to its framework we are interested in microworlds—and the specific microworld we will be working with—as sociocultural spaces with their systems of cultural significations and semiotic means of objectification in which students objectify mathematical knowledge. Microworlds activate the use of gestures, natural language, symbolic language, touching, tapping, dragging, artifacts etc. that enhance the student's sensory motor experience in space and time. We highlight the fact that the technology of a microworld would be meaningless without the culture condensed in the often highly technological nature of the SMO. A paradigmatic example of this is the software Cabri Geometry that condenses in its SMO Euclidean Geometry (Laborde, 1993).

The Microworld Mak-Trace

Mak-Trace[1] (Giorgi & Baccaglini-Frank, 2011) is an application for the iPad and iPhone in which a character can be programmed to move and draw on a grid. The grid is 10×15 and the character can only be programmed to go forwards or backwards (of the distance of one side of a square of the grid at the time) or to turn 90° clockwise or counterclockwise. The characters can be dragged on the grid with a finger to choose a starting position and then they will, by default, leave a trace mark as they move according to the commands in the programmed sequence (see Figure. 1).

The commands appear as icons that have to be dragged and placed on a vertical bar that represents the programmed sequence. Programmed sequences can be saved as MACROs and added to the possible commands to give to the character. At the moment Mak-Trace has three types of backgrounds: Free backgrounds, mazes, or backgrounds with proposed traces to program. To transition to a more "coordinate-looking" environment the characters' size can be reduced to a point and the transparency of the background increased to 100%. Mak-Trace can be recognized as a simplified version of Logo (Papert, 1980). The aim of the second author in designing Mak-Trace was to create an environment accessible to young children, or students with learning difficulties or disabilities, especially of a visuo-spatial nature, by offering an intuitive iconic programming language. Therefore we made some design choices that make this environment differ from Logo.

[1] See https://itunes.apple.com/us/app/mak-trace/id467939313?mt=8

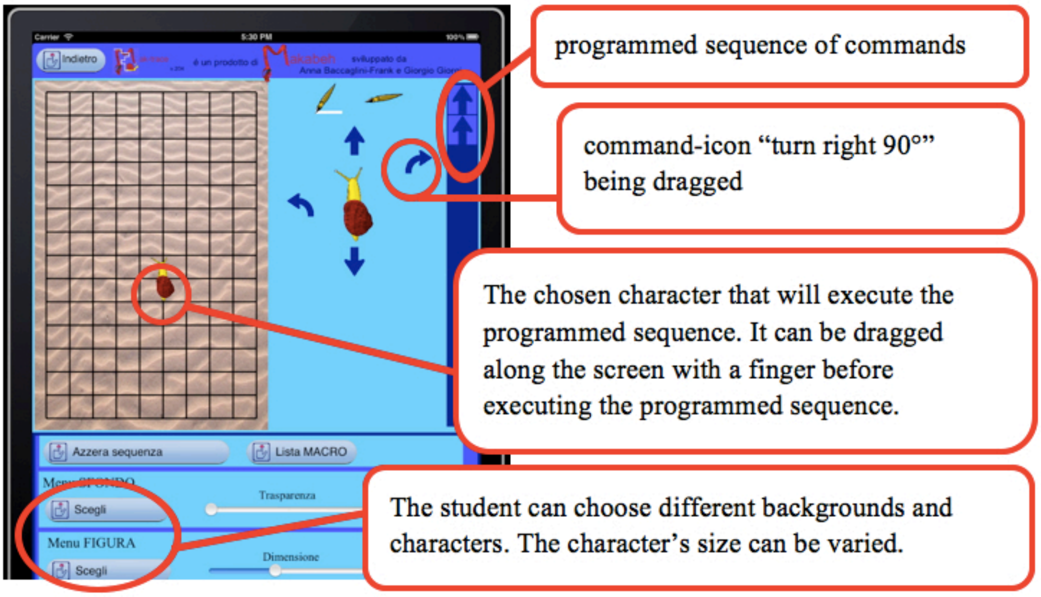

Figure 1. Main screen in Mak-Trace, where the student can program his/her character

A main difference is the fact the commands icons are made so that they can be dragged and dropped to build a sequence. This decision was made to foster the use of the commands and the sequences of commands at an embodied level, for the student programming within the environment. The fact that the command-icons can be treated as objects can make it natural to assign symbolic names to each of them in order to quickly describe a programmed sequence, orally or by writing on paper. This practice can be proposed and pursued by an educator using Mak-Trace with her students, and it may help students make use of a pre-algebraic language that can be quite useful in certain tasks involving generalization, for example when trying to find a general rule for describing inverse sequences (for an example see the following sections).

Another design choice that makes Mak-Trace different from Logo is that Mak-Trace gives no feedback in terms of movements of the character until the student touches "GO". At this point the character executes the whole list of commands in the constructed sequence. Moreover, to make a variation in the constructed sequence, the student has to go back to the programming mode: Automatically the character goes back to its original position and all trace marks are cleaned off the screen. This choice was made to foster mental planning abilities. In particular, the student has to visualize what the character will do as she is programming, and where the character will be at each step of the programmed sequence, before actually executing the sequence.

Therefore Mak-Trace can be seen a space for objectification that without the use of formal symbolic language allows the individual to attend to the mathematical concepts at all levels of generality. Its SMO cover the individual's experience and intentionality in all their complexity: time, space, perception, movement, language, the use of gestures; the student can "feel" the activity and become emotionally engaged in what s/he is doing. We expected Mak-Trace to be effective in the process of generalization because on the one hand the activity is strongly embodied and on the other hand the lack of feedback in terms of movement, during programming, obliges to visualize the path that is not perceptually present although the SMO (the iconic commands: Turn left, turn right, go forward and go backwards) act at a sensorimotor level. This environment also may foster symbolic generalization since the command-icons can be treated as objects and relations between object that reify operational invariants (rules to construct a path, a geometrical figure, to follow a path going forwards or backwards) that at even higher levels of generality may be transformed into formal language.

The Student and the Activity Sessions

The student we worked with in this study will be named Filippo. He was 15 years old and had been diagnosed with various mathematical learning disabilities including dyscalculia and severe dyslexia. From the accounts of his special education teacher (the first author), he also was not able to read maps or to give directions (however he did not have difficulty recognizing or naming his left and right hands), he had a short attention span and little—if any—interest in the activities proposed during math class. Furthermore he suffered from very low self-esteem and sense of self-efficacy. We developed a protocol so that Filippo would work with Mak-Trace when he met with his special education teacher, for five weeks, either once or twice each week. The activities he was assigned were the following: (a) describe the relationship between sequences of commands in Mak-Trace, and the movements and trace mark left by the snail; (b) program the snail to draw a square; (c) given a path, program an "inverse" sequence (first without having the snail turn around, and then having the snail turn 180° after having drawn the path in one direction) and determine a general rule for programming an inverse sequence given any sequence; and (d) complete the mazes.

In a recent paper by Baccaglini-Frank, Antonini, Robotti, and Santi (2014) excerpts from activities 1, 2, and 4 are analyzed to shed light onto the difficulties Filippo had to tackle in attempting to overcome a purely egocentric perspective (Grush, 2000). In the following section we will analyze excerpts from activity 3, which deals especially with a generalization-type task.

Analyses of Filippo's Difficulties in Generalizing

Filippo was asked to program the snail to draw a path that the teacher drew on a piece of paper (Figure 2).

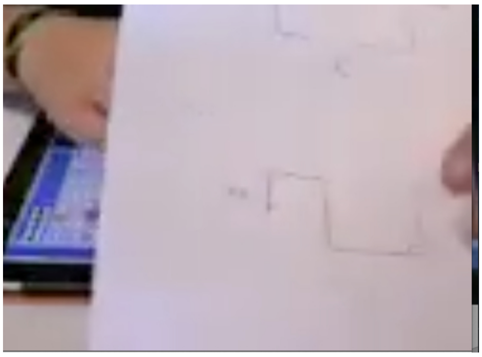

Figure 2. Path to program, drawn by the teacher on a sheet of paper

Filippo correctly programmed the snail, as shown by the trace mark left on the screen (Figure 3).

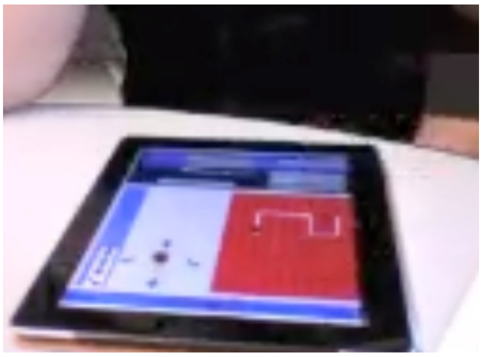

Figure 3. Filippo correctly programmed the snail to trace the path

At this point the teacher asked Filippo to dictate the sequence he had programming so that they could analyze the steps after having written them vertically on a sheet of paper. With this request, the teacher was guiding Filippo to transition to a "faster notation" (for example, "3f r 2f" as opposed to "forward, forward, forward, right, forward, forward…"). In the following excerpts T refers to the teacher and F to Filippo.

Excerpt 1

F: Up up up…

T: Slowly [writing].

F: Three up.

T: Three up.

F: Right.

T: Uh, right.

F: Up up …right…up up.

T: Wait.

F: Three up…

T: Three up.

F: Yes, because they are always, …eh, [gesture with his hand as shown in Figure 4 even though he seems to be "reading" the vertical list of commands on the screen] left.

F: Four up [holds up 4 fingers]…eh, left [makes a pointing gesture with his thumb] three up…

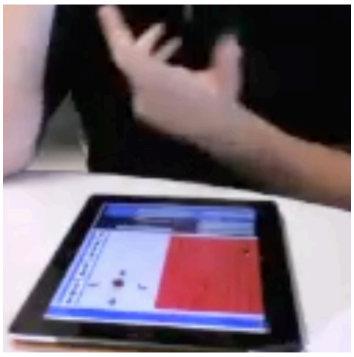

Figure 4. Filippo makes the gesture with his hand and says: "Left"

Filippo is reading from the vertical list of commands on the right of the screen. However, he still seems to feel the need of using his hand to gesture the direction of the turn. The teacher writes a vertical descending sequence of arrows, like on the iPad as Filippo dictates the commands. We notice how easily and spontaneously Filippo switches to a concise verbal notation (e.g. "three up") at this point, in what seems like an attempt at dictating the commands more efficiently to the teacher. We also note how although the teacher had insisted in using the terms "forwards, backwards, turn left, turn right" Filippo has shifted back to "up down left right" (he had used this terminology in the first two activities) as he reads of the list of commands on the vertical bar on the right. The teacher does not remark to this now, but proposes the new task on making an inverse path having the snail turn 180° after completing the path in one direction.

Excerpt 2

T: Now without looking at the picture [he puts the paper in front of Filippo on top of the iPad] you look only at the commands, is there a way, would you be able to write for me the opposite path [it: "Percorso contrario"]? Actually let's start looking at the commands [T takes away the paper and points to the list of commands on the right on the screen].

F: Like, I write them going backwards? I remember how to do it.

T: OK, do it over here [he gives F the paper and a pen]. You don't even need to look at the figure? Try it without looking. Then if you are not able to do it, we can look at it.

F: It's enough to do these things backwards [pointing to the written list of arrows].

T: OK, try.

F: [writing] I find it hard to write. [Figure 5].

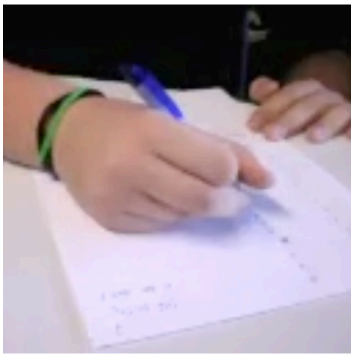

Figure 5. Filippo tries to write down the inverses of each of the commands in the sequence he dictated to the teacher

The teacher has proposed a transition to what can be seen as a pre-algebraic notation that the teacher expected might be useful when trying to find a general rule for describing inverse sequences. However this request turned out to be a bit beyond Filippo's zone of proximal development (Vygotsky, 1962) although he stayed engaged and seemed to be convinced of being able to accomplish the task. We note how at this point F refers to the arrows as "things" that can be done (and, therefore "undone" by doing them backwards).

The activity continues as Filippo counts up the steps in the two vertical sequences of arrows now written on the sheet of paper. The teacher counts a total of 22 in the new sequence and 19 in the original one, and remarks that something must be wrong.

At this point something quite interesting happens: Filippo seems unable to count up the totals, but he proceeds attempting to compare the sequences by matching up the arrows in small sets that correspond to the ones in the other column (Figure 6). He appears to be conveying meaning to the symbols on the sheet of paper and can associate small bunches of arrows to one another in a manner that appears to be *meaningful.* This shows some of his thinking and especially how he decided to start from the top of the list and reverse each arrow. To him this means substituting each arrow with its "opposite" (right with left, up with down). He even adds arrows connecting the commands in each column to the corresponding ones in the other. He counts all arrows up in groups of at most four and never all of them. On the other hand the teacher keeps counting the totals (22 and 19). The student's severe dyscalculia and dyslexia do not allow him to use numbers at a symbolic level of generalization. He counts at a factual level of generalization where perception movement, touching and pointing mediate his arithmetical thinking. He seems to be objectifying the inverse path at a factual level of generalization. However the microworld requires Filippo to work at the boundary between the factual and the contextual, and this seems to create a first source of difficulties for him. In particular, movement of the character on the screen is different from the movement experienced (or, possibly, imagined) in first person by Filippo[2].

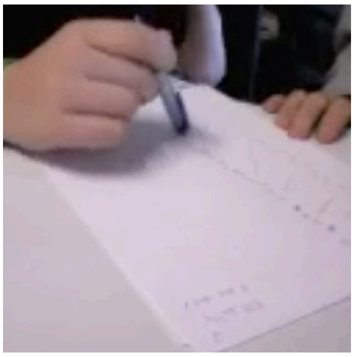

Figure 6. Filippo matches up the arrows in small sets that correspond to the ones in the other column

Filippo and the teacher correct the sets in which an arrow was missing and when what F claims to be "the inverse sequence" is programmed on the iPad, the snail traces this (Figure 7).

Figure 7. Trace mark left by the snail when Filippo programmed what he thought to be the inverse sequence

The trace mark is symmetric compared to the original path, according to a horizontal axis of symmetry. Filippo does not even realize the difference between the two trace marks. The teacher asks him to draw the two marks and only then does Filippo say.

Excerpt 3

F: Well, they are inverted.

T: Right, what is inverted?

F: This went down and this went up.

T: Right, so if you change the up with the down…[he makes a gesture showing the direction of overall tracing of the path] …here we have to instead understand how to come

[2] This difficulty has been elsewhere interpreted as a difficulty in transitioning from an egocentric to an allocentric frame of reference (Baccaglini-Frank et al., in press).

back, starting from here. [Pointing to parts of the original path traced on the sheet of paper (Figure 8)].

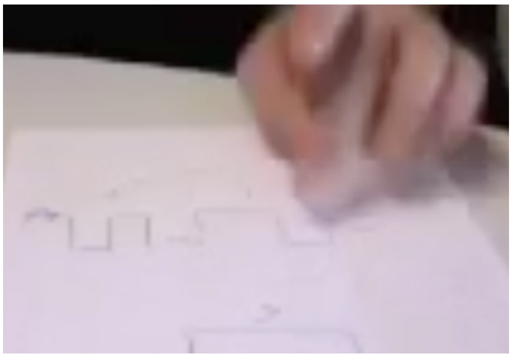

Figure 8. The teacher points to parts of the original path traced on the sheet of paper

The teacher asks Filippo to compare the two traced paths, and Filippo decides to do this by saving a Macro containing the original sequence and then programming his tentative inverse sequences. They stop working for this session, and pick up the activity again at their next meeting.

When they meet again, the teacher has set up the iPad with the Macro and he reminds Filippo of his task.

Excerpt 4

T: You need to find a *rule*, of course…a rule, like, before you transformed all the forwards into backwards and all the lefts to rights and vice versa. Instead now you should do something similar but that works. So that for every sequence, you have, starting from the initial sequence, you find the path, uhm, the commands to draw the path. I won't tell you anything. Think about it and work it out.

F: OK, profesor [He is already moving command icons onto the vertical bar]. I don't need to think about it. Trust me.

T: I trust you.

F: [Scrolls up and down on the sequence he is constructing and keeps on adding commands (Figure 9)].

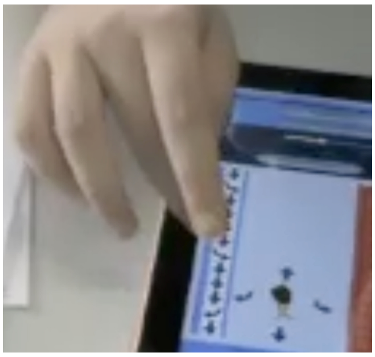

Figure 9. Filippo easily scrolls up and down to fix the sequence of commands

This is what is seen when the first three segments of the "coming back tracemark" are drawn (Figure 10). The first segment is correct, but the snail rotates to the right and continues backwards tracing a segment that is too long for the screen.

Here Filippo seems to be trying to program the snail to move along the same tracemark, going in the opposite direction as when it drew it. He seems to be treating the tracemark as a whole, a figure to be traced by the snail, and he treats it consistently with respect to the previous square drawing task, showing difficulty in deciding which direction to turn the snail in especially when it was moving towards him or away from him on vertical trajectories. We have explained this difficulty in terms of Filippo's difficulty in transitioning from an egocentric to an allocentric perspective (Baccagini-Frank et al., 2014).

Figure 10. This is what is seen when the first three segments of the "coming back tracemark" are drawn

Excerpt 5

T: Let's try to understand where you are making mistakes.

F: [Keeps the sheet of paper with the path traced in front of him, above the iPad on the desk. He decides to use trial and error, retracing the path backwards]. It will not turn out to be just the same because I do not remember the measures of the segments.

T: But we want it to turn out just the same. You need to find a schema through which given any sequence you can always find the sequence that gives the inverse sequence. It is not trivial, it is hard.

F: [Works in silence and then hits “go” (Figure 11)].

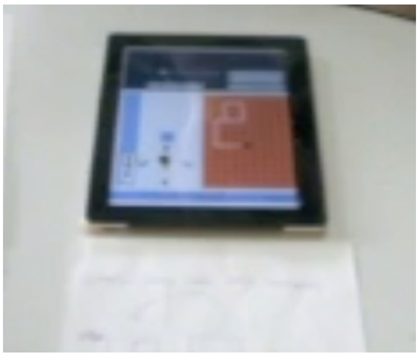

Figure 11. Trace mark left when Filippo hits “go” after his new attempt at correcting the mistake

This time the first three segments are correct, but F has inverted the third rotation (the snail is oppositely oriented and Filippo is not referring whatsoever to the initial programmed sequence, but only to the figure, so this difficulty seems to be the same as his original difficulties with the draw the square task. Here he does not seem to remember his compensatory strategies—see Baccaglini-Frank et al. (2014).

Filippo continues to program the snail after having corrected (but he does not check) the rotation. When he does check he notices he had changed the second, not the third rotation, and he tries again, this time using the rotation of the hand, one of the strategies developed to accomplish the draw the square task.

Excerpt 6

F: I understand how to do it, it’s just I get lost.

F: I’m lost,… was I here [pointing to a point on the drawn sequence on paper]?

At this point, Filippo appears to be confident in his understanding of how to solve the task, but does not seem able to devise a general strategy. Instead his understanding seems to refer to his confidence in using a trial and error strategy, also resorting to some of the compensatory strategies (such as rotating the iPad and continuing to program with it rotated as long as he “sees” the snail moving in the opposite direction) that he had come up with in previous activities (Baccaglini-Frank et al., 2014).

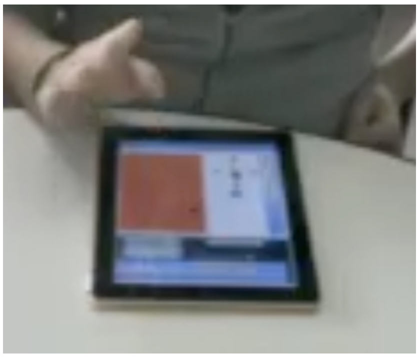

Figure 12. Filippo has remembered one of his compensatory strategies, and he uses his hand as if it were the snail

At this point, Filippo erases everything and starts over.

Excerpt 7

T: You had a good idea, you seem to be thinking about one segment. If you can understand how it works for one segment, then…

F: Then I understand everything!

T: Then you understand everything. [He tries again, but gets frustrated and bangs his hands on the table].

F: I would slap myself!

F: [He fixes the first rotation] Well up to here I did it! Now it's enough for it to turn [and he hints at a rotation with his whole body]… which is not a simple issue! [He chooses a turn and runs the program again]… And it turns! [satisfied].

Filippo now proceeds through trial and error, tracing one segment at a time and running the program each time. In total, it takes him about 20 minutes to almost complete the task for this special case It takes him an additional 10 minutes to correct the length of one of the segments of the returning trace-mark to make the marks superimposed. He is probably tired, however he succeeds in programming the snail so that it traces the path backwards.

The teacher, again, tries to introduce a condensed notation in order to elicit a symbolic comparison of the programmed sequences, hoping for Filippo to elaborate a more profound generalization based on such a comparison. The aim is to trigger a leap from the factual/contextual level to the symbolic, interpreting the arrows as objects that objectify the invariant structure of the activity and nominate them through symbols. In fact this is how mathematical objects emerge as fixed patterns of mediated activity that at higher levels of generality we subsume in language.

Excerpt 8

T: Dictate the commands to me…let's use the symbol… I will use a letter "a" [for "forwards", "*avanti*"] "i" [for "backwards", "*indietro*"] "d" [for "turn right", it. "*gira a des-*

tra”] “s” [for “turn left”, it. “*gira a sinistra*”].

F: 3i d 2i d 3i s 4i s 3i

The teacher writes down the direct and, supposedly, inverse sequences vertically. Filippo easily passes to the form 3i to say it quicker. However, only the fact of writing it and seeing the written notation seems to inhibit his spontaneity of the first time in using the compact notation. When asked to explain how he reasoned, Filippo states: “The way I do it is figuring it out on my own [*ti arrangi*] and discovering it a piece at a time.”

During the following sessions, Filippo continued to work on the same task, inverting new sequences. He always explained his way of thinking as: “I did experiments, I tried…first I tried to put the commands all reversed compared to the ones for the going path…only some parts worked.”

On the other had, the teacher kept on trying to translate the traced paths that Filippo had programmed into compact sequences written vertically descending on a piece of paper, with the objective of getting Filippo to compare the sets of symbols and figure out a general rule. Below we discuss how far Filippo seems to be able to get in this task in terms of kinds of steps he seems to be able to make (or not) in a process of generalization.

Discussion: How Far Can Filippo Get?

What we noticed is that when Filippo was asked to compare sequences (lists of commands) written in compact notation ($L_i c$, $L_j c$) he seemed to resist and when pushed, come up with rather random conjectures (e.g. “odd numbers remain the same and even numbers change” or “it’s a big mess”). This was the case even though Filippo seemed to be able to link each trace mark (T_i, T_j) to the corresponding vertical sequence of arrows on the screen (L_i, L_j). We hypothesize this to be the case because of the higher form of abstraction that each type of comparison requires. The diagram of Figure 13 may clarify in this respect.

We interpret the level of generalization involved in inverting a sequence by looking at the trace mark and imagining to go along it backwards as a factual generalization since it is bound to operational schemas within the students’ space-time embodied experience, in particular, the juggling of the egocentric and allocentric perspectives (Baccaglini-Frank et al., 2014).

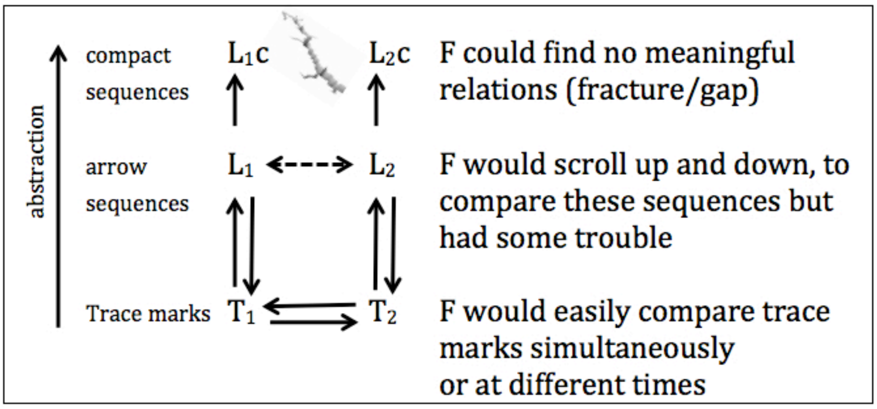

Figure 13. Diagram summarizing the increasingly difficult (for Filippo) tasks involving increasingly abstract generalization

Filippo was able to describe inverse paths as inverse sequences (sets of command icons, L_i), which he would try to compare. However he only seemed to be able to make sense of such comparisons when he would go back to the correspondence between the icons and the segments of the traced paths. In this sense we interpret these attempts as incomplete contextual generalizations.

As for the transition to comparisons of sequences in compact notation, C_i, or to a further generalization stemming from this, such as the statement of a general rule for inverting *any* sequence (what we might interpret as a symbolic generalization), Filippo did not seem to be ready to attempt such a leap on his own. However, with a lot of help from the teacher he does come to state, verbally: “They are all the same…[…] The numbers are all the same, except for two cases… right becomes left.… forwards becomes backwards […] and then I started from the end.” In this case Filippo is giving a situated account of a more general rule that with a lot of guidance he seems to have glimpsed at. Moreover, after another successful attempt at inverting a new path using trial and error, Filippo says, referring to the arrow sequences L_1 and L_2: “You have to read it backwards: You flip the commands. The last one becomes the first and they are all inverted. I could have seen this two days ago but I was not ready…”

We notice how gradually Filippo seems to be reasoning in a more abstract way: He is speaking without referring explicitly (with words) to the sequence he has just inverted, even though he is pointing to it and looking at it, and he has come to state a general working-rule for producing an inverse sequence. This is why we interpret this generalization still as a contextual generalization even though great steps have been made in abstracting away from the purely situated context.

However from the following episodes we infer that this “rule” does not actually have a general status and when Filippo is faced with a longer sequence “to invert using his rule” he resorts to his usual trial and error method and finally states: “It is a game, prof., some commands are invertible and others are not” (with this statement he is changing his mind on the invertibility of any sequence).

Concluding Remarks

We proposed a change of paradigm when investigating the teaching-learning processes of special needs students. We go beyond the distinction between normal versus special needs student and within the TO we look at the mathematical experience of students in terms of objectification taking into account the influences of disabilities on the SMO that mediate activity.

The standpoint of the TO allows us to outline the following necessary issues to consider when teaching to students.

- It is necessary to identify the SMO suitable for the specific student safeguarding his/her emotional well-being during the reflexive activity;
- generalization is not something that can be disregarded in the presence of disabilities otherwise the essence of the learning of mathematics is missed;
- it is essential to recognize the students' personal meanings and investigate the nature of gaps between these and cultural meanings we intend for our students to objectify; and
- these students bring a new voice into the communal self (Radford, 2008) that emerges from the dialogue between peers sharing the mathematical activity.

Finally, we have shown how Mak-Trace provides opportunities for a mediated reflexive activity that enhances the affective dimension of the student we worked with, improving his self-esteem and self-efficacy levels. The SMO mediate a meaningful reflexive activity because Filippo can avoid the use of symbolic language relying on his sensorimotor activity in an interplay between movement, gestures and language. Mathematical concepts appear in his space-time phenomenological experience. He generalizes at a factual-contextual level but he does not reach the symbolic one. His personal meaning is far from the cultural meaning of algebra expected by high school curriculum. This is how our framework allowed to account for the fact that the student did not seem to be able to overcome particular difficulties. Moreover, the student's experience with Mak-Trace brings a new cultural voice into the dialogue with his teacher.

Acknowledgements

We wish to thank the student, whom we have called Filippo, for accepting to work with us in this experiment, and the school IIS "E. Majorana" of San Lazzaro, Italy for enthusiastically accommodating collaborations between research and practice. We also dearly thank Anna Sfard for the insightful comments that she framed to address the issues we are concerned with.

REFERENCES

Atkinson, B. (1984). Learning disabled students and Logo. *Journal of Learning Disabilities*, *17*(8), 500-501.

Baccaglini-Frank, A., Antonini, S., Robotti, E., & Santi, G. (2014). Juggling reference frames in the microworld Mak-Trace: The case of a student with MLD. In C. Nicol, P. Liljedahl, S. Oesterle, & D. Allan. (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 2, pp. 81-88). Vancouver, Canada: PME.

Canevaro, A. (1999). *Pedagogia speciale: la riduzione dell'handicap* [Special pedagogy: the reduction of handicap]. Milan, Roma: Edizioni Bruno Mondadori.

Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the Twentieth-First Annual Meeting North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 3-26). Cuernavaca, Mexico: PME.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, *24*(2), 139-162.

Giorgi, G., & Baccaglini-Frank, A. (2011). *Mak-Trace*. [Application]. Available from: http://itunes.apple.com/it/app/maktrace/id467939313?m t=8.

Grush, R. (2000). Self, world, and space: The meaning and mechanisms of ego- and allocentric spatial representation. *Brain and Mind, 1*, 59-92.

Heyd-Metzuyanim, E. (2012). The co-construction of learning difficulties in mathematics-teacher–student interactions and their role in the development of a disabled mathematical identity. *Educational Studies in Mathematics*, *83*(3), 341-368.

Ilyenkov, E. (1977). '*The concept of the ideal', in Philosophy in the USSR: Problems of dialectical materialism*. Moscow, Russia: Progress Publishers.

Kieran, C. (1989) A perspective on algebraic thinking. In G. Vern, J. Rogalski, & M. Artigue (Eds.), *Proceedings of the 13th International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 163-171). Paris, France: Laboratoire PSYDEE.

Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics 11-16* (pp. 102-119). London, United Kingdom: Murray.

Küchemann, D., & Hoyles, C. (2009). From empirical to structural reasoning in mathematics: Tracking changes over time. In D. Stylianou, M. Blanton, & E. Knuth (Eds.), *Teaching and learning proof across the grades. A K-16 perspective* (pp. 171-191). Hillsdale, NJ: Lawrence Erlbaum.

Laborde, C. (1993). The Computer as part of the learning environment: The case of geometry. In C. Keitel & K. Ruthven (Eds.), *Learning from computers: Mathematics education and technology* (Vol. 121, pp. 48-67). Berlin and Heidelberg, Germany: NATO ASI Series, Springer-Verlag.

Leont'ev, A. N. (1978). *Activity, consciousness, and personality*. New Jersey, NJ: Prentice-Hall.

Maddux, C. (1984). Using microcomputers with the learning disabled: Will the potential be realized? *Educational Computer*, *4*(1), 31-32.

Michayluk, J. O., & Saklofske, D. H. (1988). Logo and special education. *Canadian Journal of Special Education*, *4*(1), 43-48.

Miller, P. (2009). Learning with a missing sense: What can we learn from the interaction of a deaf child with a turtle? *American Annals of the Deaf, 154*(1), 71-82.

Ministero dell'Istruzione dell'Università e della Ricerca (MIUR) (2012). *La via italiana all'inclusione scolastica, Seminario Nazionale* [Italian way to scholar inclusion, National Seminary], Roma, Italy. Accessed on May 12, 2014: http://hubmiur.pubblica.istruzione.it/web/istruzione/disabilita/inclusione-scolastica.

Papert, S. (1980). *Mindstorms*. New York, NY: Basic Books.

Radford, L. (2003). Gestures, speech, and the sprouting of signs. *Mathematical Thinking and Learning*, *5*(1), 37-70.

Radford, L. (2004). La généralisation mathématique comme processus sémiotique [Mathematical generalization as a semiotic process]. In G. Arrigo (Ed.), *Atti del convegno di didattica della matematica 2004, Alta Scuola Pedagogica* (pp. 11-27). Locarno, Switzerland: Divisione della scuola. Centro didattico cantonale.

Radford, L. (2005). Body, tool, and symbol: Semiotic reflections on cognition. In E. Simmt & B. Davis (Eds.), *Proceedings of the 2004 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 111-117). Québec, Canada: CMESG.

Radford, L (2006). The anthropology of meaning. *Educational Studies in Mathematics*, *61*(1-2), 39-65.

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 215-234). Rotterdam, The Netherlands: Sense Publishers.

Ratcliff, C., & Anderson, S.E. (2011). Reviving the turtle: Exploring the use of Logo with students with mild disabilities. *Computers in the Schools, 28*(3), 241-255.

Roth, W.-M., & Radford, L. (2011). *A cultural historical perspective on teaching and learning*. Rotterdam, The Netherlands: Sense Publishers.

Russell, S. J. (1986). But what are they learning? The dilemma of using microcomputers in special education. *Learning Disability Quarterly*, *9*(2), 100-104.

Sfard, A. (2014, April). *Conversation: Anna Baccaglini-Frank's questions about learning difficulties in mathematics.* Work presented at the Naples's Seminar at the University Federico II, Naples, Italy.

Vasu, E. S., & Tyler, D. K. (1997). A comparison of the critical thinking skills and spatial ability of fifth grade children using simulation software or Logo. *Journal of Computing in Childhood Education, 8*(4), 345-363.

Vygotsky, L. (1929). *The fundamental problems of defectology*. Retrieved online April 24, 2014, from: http://www.marxists.org/archive/vygotsky/works/1929/defectology/

Vygotsky, L. (1962). *Thought and language*. (E. Hanfmann & G. Vakar, Eds. and Trans.) Cambridge, MA: MIT Press.

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